LECTURE IV

Bi-Hamiltonian separability of Stäckel systems

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Darboux-Nijenhuis coordinates on ωN - manifold

Definition

Local coordinates (x_i, y_i) on $\omega N-$manifold are called Darboux-Nijenhuis coordinates if they are canonical with respect to $\omega = \omega_0$

$$\omega_0 = \sum_{i=1}^{n} dy_i \wedge dx_i$$

and put the recursion operator N in diagonal form

$$N = \sum_{i=1}^{n} \lambda_i \left(\frac{\partial}{\partial x_i} \otimes dx_i + \frac{\partial}{\partial y_i} \otimes dy_i \right), \qquad \lambda_i = \lambda_i(x_i, y_i).$$

- The differentials (dx_i, dy_i) span an eigenspace of N^*

$$N^* dx_i = \lambda_i dx_i, \quad N^* dy_i = \lambda_i dy_i.$$

- Distinguished DN coordinates $x_i = \lambda_i,\ y_i = \mu_i$ are called separation coordinates.

Darboux-Nijenhuis coordinates on ωN - manifold

Theorem

Bi-Lagrangian foliation is separable in (λ, μ) coordinates if control matrix F has n simple eigenvalues $(\lambda_1, ..., \lambda_n)$.

Proof.

We recoll that an n-tuple $(h_1, ..., h_n)$ of functionally independent involutive Hamiltonians is said to be separable in (λ, μ) coordinates if

$$\varphi_i(\lambda_i, \mu_i, h_1, ..., h_n) = 0, \qquad \det\left[\frac{\partial \varphi_i}{\partial h_j}\right] \neq 0.$$

Total differential is

$$\frac{\partial \varphi_i}{\partial \lambda_i} d\lambda_i + \frac{\partial \varphi_i}{\partial \mu_i} d\mu_i + \sum_{j=1}^{n} \frac{\partial \varphi_i}{\partial h_j} dh_j = 0.$$

$\square$

Darboux-Nijenhuis coordinates on ωN - manifold

Apply N^*

$$\frac{\partial \varphi_i}{\partial \lambda_i}\lambda_i d\lambda_i + \frac{\partial \varphi_i}{\partial \mu_i}\lambda_i d\mu_i + \sum_{j=1}^{n} \frac{\partial \varphi_i}{\partial h_j} N^* dh_j = 0,$$

so

$$\sum_{j=1}^{n} \frac{\partial \varphi_i}{\partial h_j} N^* dh_j = -\lambda_i \left(\frac{\partial \varphi_i}{\partial \lambda_i} d\lambda_i + \frac{\partial \varphi_i}{\partial \mu_i} d\mu_i \right) = \lambda_i \sum_{j=1}^{n} \frac{\partial \varphi_i}{\partial h_j} dh_j,$$

or in matrix form

$$JN^* dh = \Lambda J dh,$$

where

$$J^i_j = \frac{\partial \varphi_i}{\partial h_j}, \quad dh = (dh_1, ..., dh_n)^T, \quad \Lambda = diag(\lambda_1, ..., \lambda_n).$$

Therefore

$$N^* dh_i = \sum_{j=1}^{n} F_{ij}\ dh_j, \qquad F = J^{-1} \Lambda J, \tag{4.1}$$

so F is a control matrix with eigenvalues $(\lambda_1, ..., \lambda_n)$.

Stäckel separability

- Stäckel separation relations

$$\sum_{k=1}^{n} S_i^k(\lambda_i,\mu_i)h_k = \psi_i(\lambda_i,\mu_i), \qquad i=1,\dots,n,$$

so $J_j^i = S_j^i$, S - Stäckel matrix $\Longrightarrow F = S^{-1}\Lambda S$.

- Bare Hamiltonians

$$\sum_{k=1}^{m} \varphi_i^k(\lambda_i,\mu_i)E^{(k)}(\lambda_i,n_k) = \chi_i(\lambda_i,\mu_i), \qquad i=1,\dots,n. \qquad (4.2)$$

- Generalized separable potentials

$$\sum_{k=1}^{m} \varphi_i^k(\lambda_i,\mu_i)\left[\lambda_i^{r_s}\delta_k^s + V^{(k,s,r_s)}(\lambda_i,n_k)\right] = 0, \qquad i=1,\dots,n, \quad (4.3)$$

- so

$$H_i^{(k)} = E_i^{(k)} + \sum_{s=1}^{m} V_i^{(k,s,r_s)}.$$

Stäckel separability

- Trivial potentials

$$V_i^{(k,s,n_s-j)} = -\delta_{sk}\delta_j^i, \qquad i,j = 1,\dots,n_s.$$

- In particular

$$V_i^{(k,s,0)} = -\delta_{sk}\delta_{n_s}^i.$$

- Define vectors

$$V^{(s,r)} = \left(V_1^{(1,s,r)},\dots,V_{n_1}^{(1,s,r)},\dots,V_{n_m}^{(m,s,r)}\right)^T,$$
$$\varphi^{(k)} = (\varphi_1^{(k)},\dots,\varphi_n^{(k)})^T, \qquad s,k = 1,\dots,m.$$

- Then, separation relations for generalized potentials (4.3) take the matrix form

$$\Lambda^r\varphi^{(s)} + SV^{(s,r)} = 0, \qquad s = 1,\dots,m.$$

Stäckel separability

Lemma

$$V^{(s,r+1)} = FV^{(s,r)}, \qquad s = 1, \ldots, m.$$

Proof.

$$\Lambda^{r+1}\varphi^{(s)} + SV^{(s,r+1)} = 0$$

$$\Downarrow$$

$$V^{(s,r+1)} = -S^{-1}\Lambda^{r+1}\varphi^{(s)} = -S^{-1}\Lambda\Lambda^{r}\varphi^{(s)} = S^{-1}\Lambda SV^{(s,r)} = FV^{(s,r)}.$$

$\square$

- As Λ and S are invertible so do F.

Stäckel separability

- We have m hierarchies of basic separable potentials

$$V^{(s,r)} = F^r V^{(s,0)}, \qquad s = 1, \dots, m, \quad r \in \mathbb{Z}.$$

- It means that

$$F = \left[-V^{(s,n_s-j)} \right], \qquad s = 1, \dots, m, \quad j = 0, \dots, n_s - 1.$$

- Nontrivial columns: $V^{(s,n_s)}$ - first nontrivial potentials of each hierarchy.

Stäckel separability

- So, for Stäckel Hamiltonians

$$N^* dh_i = \sum_{j=1}^{n} F_{ij}\ dh_j$$

$$\Updownarrow$$

$$N^* dh_i^{(k)} = dh_{i+1}^{(k)} - \sum_{j=1}^{m} \alpha_{ij}^{(k)} dh_1^{(j)}, \qquad \alpha_{ij}^{(k)} = V_i^{(k,j,n_j)}$$

$$\Downarrow$$

$$\pi_1 dh_i^{(k)} = \pi_0 \left(dh_{i+1}^{(k)} - \sum_{j=1}^{m} \alpha_{ij}^{(k)} dh_1^{(j)} \right)$$

- where

$$\pi_0 = \sum_i \frac{\partial}{\partial \lambda_i} \wedge \frac{\partial}{\partial \mu_i}, \quad \pi_1 = \sum_i \lambda_i \frac{\partial}{\partial \lambda_i} \wedge \frac{\partial}{\partial \mu_i}.$$

Toward bi-Hamiltonian construction

- Extended $2n$-dimensional phase space $M \to \mathcal{M} = M \times \mathbb{R}^m$, with additional coordinates c_i, $i = 1, ..., m$ and extended Poisson structures

$$\pi_0 \to \Pi_0 = \begin{bmatrix} \pi_0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \pi_1 \to \Pi_D = \begin{bmatrix} \pi_1 & 0 \\ 0 & 0 \end{bmatrix}$$

with $Ker\Pi_0 = Ker\Pi_{1D} = Sp\{dc_i\}$.

- Take $\mathcal{Z} = Sp\{\frac{\partial}{\partial c_i}\}$. Obviously

$$Z_i(c_j) = \delta_{ij}, \quad [Z_i, Z_j] = 0, \qquad L_{Z_i}\Pi_0 = 0$$

- and hence

$$Z_j(H_i^{(k)}) = \alpha_{ij}^{(k)} = V_i^{(k,j,n_j)} \Longrightarrow H_i^{(k)} = h_i^{(k)} + \sum_{j=1}^{m} V_i^{(k,j,n_j)} c_j.$$

Toward bi-Hamiltonian construction

- Moreover, also

$$\Pi_1 dH_i^{(k)} = \Pi_0 \left(dH_{i+1}^{(k)} - \sum_{j=1}^{m} \alpha_{ij}^{(k)} dH_1^{(j)} \right)$$

as for any separable potential $V^{(s,r)}$

$$N^* dV^{(s,r)} = F dV^{(s,r)}.$$

- So, according to construction presented in Lecture III, Hamiltonians $(H_0^{(1)}, ..., H_{n_m}^{(m)})$, where $H_0^{(k)} \equiv c_k$, form a bi-Hamiltonian chains

$$\begin{aligned} \Pi_0 dH_0^{(k)} &= 0 \\ \Pi_0 dH_1^{(k)} &= X_1^{(k)} = \Pi_1 dH_0^{(k)} \\ &\vdots \\ \Pi_0 dH_{n_k}^{(k)} &= X_{n_k}^{(k)} = \Pi_1 dH_{n_k - 1}^{(k)} \\ 0 &= \Pi_1 dH_{n_k}^{(k)} \end{aligned}$$

Toward bi-Hamiltonian construction

- where

$$\Pi_1 = \Pi_D + \sum_{j=1}^{m} X_1^{(j)} \wedge Z_j = \sum_{i=1}^{n} \lambda_i \frac{\partial}{\partial \lambda_i} \wedge \frac{\partial}{\partial \mu_i} + \sum_{j=1}^{m} X_1^{(j)} \wedge \frac{\partial}{\partial c_j},$$

- with separation relations

$$\sum_{k=1}^{m} \varphi_i^k(\lambda_i, \mu_i) \left[\lambda_i^{r_s} \delta_k^s + H^{(k)}(\lambda_i, n_k) \right] = \chi_i(\lambda_i, \mu_i),$$

$$H^{(k)}(\lambda, n_k) = \sum_{j=0}^{n_k} H_j^{(k)} \lambda^{n_k - j} = c_k \lambda^{n_k} + h^{(k)}(\lambda, n_k).$$

Benenti class

- Separation relations for Benenti class of Stäckel systems $(m=1, s=1)$

$$\lambda_i^r + h_1\lambda_i^{n-1} + \ldots + h_n = \frac{1}{2} f_i(\lambda_i)\mu_i^2, \qquad i = 1, \ldots, n$$

$$\Updownarrow \; h_j = E_j + V_j^{(r)}$$

$$E_1\lambda_i^{n-1} + \ldots + E_n = \frac{1}{2} f_i(\lambda_i)\mu_i^2 \qquad \text{geodesic Hamiltonians}$$

$$\lambda_i^r + V_1^{(r)}\lambda_i^{n-1} + \ldots + V_n^{(r)} = 0 \qquad \text{separable potentials}$$

- Control matrix (recursion matrix)

$$F = \begin{bmatrix} -\rho_1 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ \vdots & 0 & 0 & 1 \\ -\rho_n & 0 & 0 & 0 \end{bmatrix}, \qquad \rho_i - \text{ Viete polynomials}$$

Benenti class

- Quasi-bi-Hamiltonian chain

$$\pi_1 dh_i = \pi_0(dh_{i+1} - \rho_i dh_1), \qquad i = 1, ..., n$$

- and basic separable potentials

$$V^{(k)} = F^k V^{(0)}, \qquad k \in \mathbb{Z}$$

$$V^{(k)} = (V_1^{(k)}, ..., V_n^{(k)})^T, \qquad V^{(0)} = (0, ..., 0, -1)^T.$$

- The extention to bi-Hamiltonian chain on $\mathcal{M} = M \times \mathbb{R}$

$$\begin{aligned}
\Pi_0 dH_0 &= 0 \\
\Pi_0 dH_1 &= X_1 = \Pi_1 dH_0 \\
&\vdots \\
\Pi_0 dH_{n_k} &= X_n = \Pi_1 dH_n \\
0 &= \Pi_1 dH_n
\end{aligned}$$

Benenti class

- where

$$H_i = h_i + \rho_i c, \qquad i = 1, ..., n, \quad H_0 = c, \quad \rho_i = V_i^{(n)}$$

  and

$$\Pi_0 = \begin{bmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Pi_0 = \begin{bmatrix} 0 & \Lambda & X_1 \\ -\Lambda & 0 & \\ -X_1^T & & 0 \end{bmatrix}.$$

- Separation relations for H_i are

$$\lambda_i^r + c\lambda_i^n + H_1\lambda_i^{n-1} + ... + H_n = \frac{1}{2} f_i(\lambda_i)\mu_i^2, \qquad i = 1, ..., n.$$

Bi-Hamiltonian representation of extended Henon-Heiles system in Euclidian coordinates

$$
\begin{aligned}
H_0 &= c \\
H_1 &= \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + q_1^3 + \frac{1}{2}q_1 q_2^2 - q_1 c \equiv H \\
H_2 &= \frac{1}{2}q_2 p_1 p_2 - \frac{1}{2}q_1 p_2^2 + \frac{1}{4}q_1^2 q_2^2 + \frac{1}{16}q_2^2 - \frac{1}{4}q_2^2 c,
\end{aligned}
$$

$$
\begin{aligned}
\Pi_0 dH_0 &= 0 \\
\Pi_0 dH_1 &= X_1 = \Pi_1 dH_0 \\
\Pi_0 dH_2 &= X_2 = \Pi_1 dH_1 \\
0 &= \Pi_1 dH_2
\end{aligned}
$$

Henon-Heiles system

- where

$$\Pi_1 = \left[\begin{array}{ccccc} 0 & 0 & q_1 & \frac{1}{2}q_2 & \\ 0 & 0 & \frac{1}{2}q_2 & 0 & X_1 \\ -q_1 & -\frac{1}{2}q_2 & 0 & \frac{1}{2}p_2 & \\ -\frac{1}{2}q_2 & 0 & -\frac{1}{2}p_2 & 0 & \\ & -X_1^T & & & 0 \end{array}\right] \quad X_1 = \left[\begin{array}{c} p_1 \\ p_2 \\ -3q_1^2 - \frac{1}{2}q_2^2 + c \\ -q_1 q_2 \\ 0 \end{array}\right]$$

- Transversal distribution $\mathcal{Z}$, such that Π_1 is $\mathcal{Z}$-invariant, is one-dimensional with $Z = \frac{\partial}{\partial c}$, hence $\Pi_D = \Pi_1 - X_1 \wedge Z$ so

$$\pi_1 = \left[\begin{array}{cccc} 0 & 0 & q_1 & \frac{1}{2}q_2 \\ 0 & 0 & \frac{1}{2}q_2 & 0 \\ -q_1 & -\frac{1}{2}q_2 & 0 & \frac{1}{2}p_2 \\ -\frac{1}{2}q_2 & 0 & -\frac{1}{2}p_2 & 0 \end{array}\right].$$

- Separation coordinates λ_1 and λ_2 are eigenvalues of the recursion operator $N = \pi_1 \pi_0^{-1}$,

Henon-Heiles system

- hence

$$\sqrt{(\lambda I - N)} = 0 \Longrightarrow q_1 = \lambda_1 + \lambda_2,\ \ q_2 = 2\sqrt{-\lambda_1\lambda_2}$$

$$\Downarrow$$

$$p_1 = \frac{\lambda_1\mu_1}{\lambda_1 - \lambda_2} + \frac{\lambda_2\mu_2}{\lambda_2 - \lambda_1}, \qquad p_1 = \sqrt{-\lambda_1\lambda_2}\left(\frac{\mu_1}{\lambda_1 - \lambda_2} + \frac{\mu_2}{\lambda_2 - \lambda_1}\right).$$

- Separation relations are

$$c\lambda_i^2 + H_1\lambda_i + H_2 = \frac{1}{2}\lambda_i\mu_i^2 + \lambda_i^4, \qquad i = 1, 2$$

- and control matrix

$$F = \begin{bmatrix} q_1 & 1 \\ \frac{1}{4}q_2^2 & 0 \end{bmatrix}.$$

Henon-Heiles system

- In fact we have infinite family of separable bi-Hamiltonian systems in Euclidian coordinates

$$H_1 = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + V_1^{(k)} - q_1 c, \quad H_2 = \frac{1}{2}q_2 p_1 p_2 - \frac{1}{2}q_1 p_2^2 + V_2^{(k)} - \frac{1}{4}q_2^2 c,$$

where

$$\begin{bmatrix} V_1^{(k)} \\ V_2^{(k)} \end{bmatrix} = \begin{bmatrix} q_1 & 1 \\ \frac{1}{4}q_2^2 & 0 \end{bmatrix}^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad k \in \mathbb{Z}$$

- with separation relations

$$c\lambda_i^2 + H_1\lambda_i + H_2 = \frac{1}{2}\lambda_i \mu_i^2 + \lambda_i^k, \qquad i = 1, 2.$$

- For example

$$V^{(5)} = \begin{bmatrix} q_1^4 + \frac{3}{4}q_1^2 q_2^2 + \frac{1}{16}q_2^4 \\ \frac{1}{4}q_1^3 q_2^2 + \frac{1}{8}q_1 q_2^4 \end{bmatrix}, \qquad V^{(-1)} = \begin{bmatrix} 4q_2^{-2} \\ -4q_1 q_2^{-2} \end{bmatrix}.$$